

Quantum field theory for discrepancies II: $1/N$ corrections using fermions

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Abstract

We calculate the $1/N$ corrections to the probability distributions of quadratic discrepancies for sets of N random points. This is achieved by the introduction of fermionic variables. We give the diagrammatic expansion up to and including the second order in $1/N$. For some discrepancies, we give the explicit expansion to first order.

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1 Introduction

DISCREPANCIES are measures of non-uniformity of point sets that play an important rôle in the Quasi-Monte Carlo method of numerical integration [1]. A certain class of these discrepancies, the so called *quadratic discrepancies*, can be defined as an average-case complexity over a class of functions, the *problem class* [2, 3, 4]. In a number of publications [5, 6, 7] the problem of calculating the probability distribution of discrepancies for sets of N truly random points has been tackled. One of the results was the calculation of the asymptotic distributions in the limit of infinite N .

In [4], we introduced techniques from quantum field theory (QFT) to calculate the moment generating function G of the probability distribution, suitable to calculate it as a series expansion in $1/N$. In this paper, we extend the formalism by the introduction of fermions, and give the explicit diagrammatic expansion of $\log G$ up to and including $\mathcal{O}(1/N^2)$. For the Lego discrepancy, the L_2^* -discrepancy in one dimension and the Fourier diaphony in one dimension, we give the explicit $1/N$ correction.

2 The formalism

We start with a short repetition of the formalism of [4] and continue with the introduction of some new tools.

2.1 Quadratic discrepancies and path integrals

We shall always take the integration region to be the s -dimensional unit hypercube $\mathbf{K} = [0, 1]^s$. The point set X_N consists of N points $x_k \in \mathbf{K}$, $k = 1, 2, \dots, N$. Defined as an average-case complexity on the problem class of functions $\phi : \mathbf{K} \mapsto \mathbf{R}$ with measure μ , a discrepancy D_N of the point set X_N is given by

$$D_N = N \int \eta_N^2[\phi] d\mu[\phi] \quad , \quad \eta_N[\phi] = \frac{1}{N} \sum_{k=1}^N \phi(x_k) - \int_{\mathbf{K}} \phi(x) dx \quad . \quad (1)$$

It is the quadratic integration error, averaged over the problem class. The probability density H of the discrepancy as a random variable of random point sets is calculated as the inverse Laplace transform of the moment generating function G :

$$H(D_N = t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{-zt} G(z) dz \quad , \quad G(z) = \mathbf{E}[e^{zD_N}] \quad . \quad (2)$$

Here, \mathbf{E} denotes the expectation value of a random variable.

The measure μ is assumed to be Gaussian and in [4] it is shown that the generating function is therefore given by

$$G(z) = \int \left(\int_{\mathbf{K}} e^{g[\phi(x) - \int_{\mathbf{K}} \phi(y) dy]} dx \right)^N d\mu[\phi] \quad , \quad g = \sqrt{\frac{2z}{N}} \quad . \quad (3)$$

In [4] we suggested to put the N^{th} power in the exponential and interpret $G(z)$ as an Euclidean path integral

$$G(z) = \int \mathcal{D}\phi \exp(-S[\phi]) \quad (4)$$

with an action

$$S[\phi] = \frac{1}{2} \int_{\mathbf{K}^2} \phi(x) \Lambda(x, y) \phi(y) dx dy - N \log \left(\int_{\mathbf{K}} e^{g[\phi(x) - \int_{\mathbf{K}} \phi(y) dy]} dx \right) , \quad (5)$$

where Λ is the symmetric linear operator with boundary conditions which is the inverse of the two-point Green function under the measure μ :

$$\int_{\mathbf{K}} \Lambda(x_1, y) \mathcal{C}(y, x_2) dy = \delta(x_1 - x_2) \quad , \quad \mathcal{C}(y, x_2) = \int \phi(y) \phi(x_2) d\mu[\phi] . \quad (6)$$

The “infinitesimal volume element” $\mathcal{D}\phi$ can be seen as defined by the rule that $d\mu[\phi] = \mathcal{D}\phi \exp(-S_0)$, where S_0 is the action with $z = 0$. For notational convenience we put $\mathcal{D}\phi$ to the left of the exponential.

One of the features of this formalism is that the action has a *gauge freedom*; a *global translation* $\Theta_c : \phi(x) \mapsto \phi(x) + c$, $c \in \mathbf{R}$ leaves $\eta_N[\phi]$ and *a fortiori* $G(z)$ invariant, and results in a change of the action at most linear in ϕ :

$$S[\Theta_c \phi] = S[\phi] + \alpha c \chi[\phi] + \frac{1}{2} \alpha c^2 \quad \text{with} \quad \chi[\phi_1 + \phi_2] = \chi[\phi_1] + \chi[\phi_2] . \quad (7)$$

As a result of this, the path integral can be generally written as

$$G(z) = \frac{1}{I[F]} \sqrt{\frac{2\pi}{\alpha}} \int \mathcal{D}\phi \exp(-F(\xi[\phi]) - S_\Theta[\phi]) , \quad S_\Theta[\phi] = S[\phi] - \frac{1}{2} \alpha \chi[\phi]^2 , \quad (8)$$

where F is restricted such that

$$I[F] \equiv \int_{-\infty}^{\infty} \exp(-F(c)) dc \quad (9)$$

exists, and ξ is only restricted such that $\xi[\Theta_c \phi] = \xi[\phi] + c$. It is, for example, possible to take $\chi[\phi] = \int_{\mathbf{K}} \phi(x) dx$ and $F(\chi) = M \chi^2$ with $M \rightarrow \infty$. In this gauge, $\chi[\phi]$ will vanish from the action, which is reflected in a two-point function that integrates to zero with respect to each of its variables. We shall refer to this gauge as the *Landau* gauge.

Clearly, the first equation of (6) cannot be satisfied in the Landau gauge, because then \mathcal{C} integrates to zero ¹. To see what happens, we assume that the problem class is a vector space with a basis of at least square integrable functions $\{u_n\}$, so that a member ϕ of the problem class can be written as

$$\phi(x) = \sum_n \phi_n u_n(x) , \quad \phi_n \in \mathbf{R} , \quad (10)$$

¹There is a misprint in Eq. (73) of [4] with respect to this, where $\delta(x - y)$ should be replaced by $\delta(x - y) - 1$.

and that the Gaussian measure on this function space is defined by

$$d\mu[\phi] = \prod_n \frac{\exp(-\phi_n^2/2\sigma_n^2)}{\sqrt{2\pi\sigma_n^2}} d\phi_n \ , \quad \sigma_n \in \mathbf{R} \ . \quad (11)$$

For the measure to be suitably defined, the strengths σ_n have to satisfy certain restrictions which can be translated into the requirement that $E[D_N]$ exists. They are the inverse of the eigenvalues of \mathcal{C} and the basis consists of the eigenfunctions. Therefore, \mathcal{C} and Λ can be expressed in terms of the basis:

$$\mathcal{C}(x_1, x_2) = \sum_n \sigma_n^2 u_n(x_1) u_n(x_2) \ , \quad \Lambda(x_1, x_2) = \sum_n \frac{1}{\sigma_n^2} u_n(x_1) u_n(x_2) \ , \quad (12)$$

where the last equation holds if the basis is orthonormal. The boundary conditions satisfied by \mathcal{C} and Λ are those satisfied by the basis functions. With different gauges come different bases and strengths. We call a gauge in which the basis is orthonormal a *Feynman* gauge. If the Landau gauge is used, in which $\int_{\mathbf{K}} \phi(x) dx = 0$, then the basis functions have to integrate to zero:

$$\int_{\mathbf{K}} u_n^{(L)}(x) dx = 0 \quad \forall n \ , \quad (13)$$

where the label L indicates the Landau gauge. This means that the basis cannot be “complete” in the sense that $\sum_n u_n(x) u_n(y) = \delta(x - y)$, but that we have

$$\sum_n u_n^{(L)}(x) u_n^{(L)}(y) = \delta(x - y) - 1 \ . \quad (14)$$

The zero mode is isolated and integrated out of the path integral. We want to stress that the gauge freedom is something that comes from the original measure μ , and that the Landau gauge exists for every quadratic discrepancy. It is a result of the fact that an integration error is the same for integrands that differ only by a constant.

From now on we will denote the two-point function in the Landau gauge by \mathcal{B} . It satisfies

$$\int_{\mathbf{K}} \Lambda^{(L)}(x_1, y) \mathcal{B}(y, x_2) dy = \delta(x_1 - x_2) - 1 \ , \quad (15)$$

and the discrepancy can then be written as

$$D_N = \frac{1}{N} \sum_{k,l=1}^N \mathcal{B}(x_k, x_l) \ . \quad (16)$$

2.2 Fermions as tools to calculate the $1/N$ corrections

In [4] we suggested to make a straight forward expansion in $1/N$ of $\exp(-S)$ to calculate G perturbatively. This way, however, the calculation of the perturbation series becomes very cumbersome, and the reason for this is the following. We want to use the fact that an expansion in $1/N$ corresponds to an expansion around $\phi = 0$ of the part of the action that is non-quadratic in ϕ . The subsequent terms in the expansions are therefore proportional to moments of a Gaussian measure, and can be calculated using diagrams (cf. [10]). These diagrams, the *Feynman diagrams*, consist of lines representing two-point functions and vertices representing convolutions of two-point functions. Because the action is non-local, i.e. it cannot be written as a single integral over a Lagrangian density because of the logarithm in Eq. (5), the total path integral, thus the total sum of all diagrams, cannot be seen as the exponential of all *connected* diagrams, and it is this that makes the calculations difficult.

In order to circumvent this obstacle, we introduce $2N$ Grassmann variables $\bar{\psi}_i$ and ψ_i , $i = 1, 2, \dots, N$. They all anti-commute with each other and commute with complex numbers:

$$\bar{\psi}_i \bar{\psi}_j + \bar{\psi}_j \bar{\psi}_i = 0 \quad , \quad \bar{\psi}_i \psi_j + \psi_j \bar{\psi}_i = 0 \quad , \quad \psi_i \psi_j + \psi_j \psi_i = 0 \quad i, j = 1, 2, \dots, N \quad (17)$$

$$c\bar{\psi}_i - \bar{\psi}_i c = 0 \quad , \quad c\psi_i - \psi_i c = 0 \quad i = 1, 2, \dots, N \quad , \quad c \in \mathbf{C} \quad . \quad (18)$$

Now we use the well known Gaussian integration rules for Grassmann variables to write the N^{th} power in Eq. (3) as an exponent and get

$$G(z) = \int \mathcal{D}\phi D\bar{\psi} D\psi \exp(-S[\phi, \bar{\psi}, \psi]) \quad , \quad (19)$$

$$S[\phi, \bar{\psi}, \psi] = \frac{1}{2} \int_{\mathbf{K}^2} \phi(x) \Lambda(x, y) \phi(y) dx dy + \int_{\mathbf{K}} e^{g[\phi(x) - \int_{\mathbf{K}} \phi(y) dy]} dx \sum_{i=1}^N \bar{\psi}_i \psi_i \quad , \quad (20)$$

where we introduced the notational shorthand

$$D\bar{\psi} = d\bar{\psi}_1 d\bar{\psi}_2 \cdots d\bar{\psi}_N \quad , \quad D\psi = d\psi_1 d\psi_2 \cdots d\psi_N \quad . \quad (21)$$

Notice that this action contains the same gauge freedom, so that the action becomes completely local if the Landau gauge is used. We have to introduce the “fermion fields” to achieve this, but for calculating the perturbation expansion they are much easier to handle than the logarithmic potential. From the action (20) we obtain the Feynman rules. To calculate a term in the $1/N$ -expansion of G , the contribution of all diagrams that can be drawn using the Feynman rules and carry the right power of $1/N$ has to be calculated.

Because in this paper we want to calculate the path integral itself, and no correlation functions, we only have to consider *vacuum* diagrams, i.e., diagrams without external legs. Furthermore, we will always use the Landau gauge, because then action is local, so that we only need to calculate *connected* diagrams. The whole generating function is the

3.1 The zeroth order

The contribution to the zeroth order in $1/N$ can only come from diagrams in which the power of $1/N$ coming from the vertices cancels the power of N , coming from the fermion loops. This only happens in diagrams with vertices with two bosonic legs only, and in which the fermion lines begin and end on the same vertex. To write down their contribution, we introduce the two-point functions \mathcal{B}_p , $p = 1, 2, \dots$, defined by

$$\mathcal{B}_1(x_1, x_2) = \mathcal{B}(x_1, x_2) \quad , \quad \mathcal{B}_{p+1}(x_1, x_2) = \int_{\mathbf{K}} \mathcal{B}_p(x_1, y) \mathcal{B}(y, x_2) dy \quad . \quad (28)$$

The zeroth order term is given by

$$\frac{1}{2} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{6} \text{diagram} + \dots = \sum_{p=1}^{\infty} \frac{(Ng^2)^p}{2p} \int_{\mathbf{K}} \mathcal{B}_p(x, x) dx \quad . \quad (29)$$

The factor $1/2p$ is the symmetry factor of this type of diagram with p fermion “leaves”. If we substitute $g = \sqrt{2z/N}$ in this expression, we find exactly the result of Eq. (21) in [8]. If we use the spectral representation of \mathcal{B} and assume that the basis functions are orthonormal in the Landau gauge, we get

$$W_0(z) = -\frac{1}{2} \sum_n \log(1 - 2z\sigma_{L,n}^2) \quad , \quad (30)$$

where L indicates the Landau gauge. This expression is the same as Eq. (68) in [5].

3.2 The first order

As we have seen before, bosonic two-point vertices with a closed single fermion line contribute with a factor $2z$, and without any dependence on N . Therefore, it is useful to introduce the following effective vertex

$$\text{diagram} = \text{diagram} = Ng^p \times \text{convolution} \quad , \quad (31)$$

and the following *dressed* boson propagator

$$\begin{aligned} x \text{---} y &= x \text{-----} y + x \text{---}\bullet\text{---} y + x \text{---}\bullet\text{---}\bullet\text{---} y \\ &\quad + x \text{---}\bullet\text{---}\bullet\text{---}\bullet\text{---} y + \dots \\ &= \sum_{p=1}^{\infty} (2z)^{p-1} \mathcal{B}_p(x, y) \quad . \end{aligned} \quad (32)$$

If we assume that the basis is orthonormal in the Landau gauge, we can write

$$x \text{ --- } y = \sum_n \frac{\sigma_{L,n}^2}{1 - 2z\sigma_{L,n}^2} u_n^{(L)}(x) u_n^{(L)}(y) , \quad (33)$$

which is, apart of a factor $2z$, the same expression as in Eq. (67) in [5]. The dressed propagator is equal to the propagator in the Landau gauge as we defined it in Eq. (24) in [4]:

$$x \text{ --- } y = \mathcal{G}_z^{(L)}(x, y) . \quad (34)$$

This is easy to see, because it satisfies

$$\int_{\mathbf{K}} [A^{(L)}(x_1, y) - 2z\delta(x_1 - y) + 2z] \sum_{p=1}^{\infty} (2z)^{p-1} \mathcal{B}_p(x, y) dy = \delta(x_1 - x_2) - 1 , \quad (35)$$

just like $\mathcal{G}_z^{(L)}$ by definition. Notice that $\mathcal{G}_z^{(L)}$ and \mathcal{B} satisfy the relation

$$\lim_{z \rightarrow 0} \mathcal{G}_z^{(L)}(x, y) = \mathcal{G}_{z=0}^{(L)}(x, y) = \mathcal{B}(x, y) \quad \forall x, y \in \mathbf{K} . \quad (36)$$

Furthermore, notice that $\mathcal{G}_z^{(L)}$ and W_0 satisfy

$$\frac{\partial}{\partial z} W_0(z) = \int_{\mathbf{K}} \mathcal{G}_z^{(L)}(x, x) dx , \quad (37)$$

and that this relation determines W_0 uniquely, because we know that $W_0(0)$ has to be equal to 0 in order for the asymptotic probability distribution to be normalized to 1. From now on, we will omit the label L.

The first order term in the expansion of $W(z)$ is

$$\frac{1}{N} W_1(z) = \frac{1}{8} \text{ (figure 8) } + \frac{1}{8} \text{ (figure 8 with arrows) } + \frac{1}{4} \text{ (figure 8 with arrows and a circle) } + \frac{1}{8} \text{ (figure 8 with a circle) } + \frac{1}{12} \text{ (figure 8 with a circle and a dot) } , \quad (38)$$

or, more explicitly,

$$\begin{aligned} W_1(z) = & \frac{z^2}{2} \int_{\mathbf{K}} \mathcal{G}_z(x, x)^2 dx - \frac{z^2}{2} \left(\int_{\mathbf{K}} \mathcal{G}_z(x, x) dx \right)^2 - z^2 \int_{\mathbf{K}^2} \mathcal{G}_z(x, y)^2 dx dy \\ & + z^3 \int_{\mathbf{K}^2} \mathcal{G}_z(x, x) \mathcal{G}_z(x, y) \mathcal{G}_z(y, y) dx dy + \frac{2z^3}{3} \int_{\mathbf{K}^2} \mathcal{G}_z(x, y)^3 dx dy . \end{aligned} \quad (39)$$

3.3 The second order

The second order term in the expansion of $W(z)$ is denoted by $\frac{1}{N^2} W_2(z)$ and is given by

$$\begin{aligned} & \frac{1}{48} \text{ (figure 8 with a circle) } + \frac{1}{48} \text{ (figure 8 with a circle and a dot) } + \frac{1}{16} \text{ (figure 8) } + \frac{1}{12} \text{ (figure 8 with a circle) } + \frac{1}{24} \text{ (figure 8 with a circle and a dot) } + \frac{1}{16} \text{ (figure 8 with a circle and a dot) } \\ & + \frac{1}{8} \text{ (figure 8 with a circle) } + \frac{1}{8} \text{ (figure 8 with a circle and a dot) } + \frac{1}{8} \text{ (figure 8 with a circle and a dot) } + \frac{1}{16} \text{ (figure 8 with a circle and a dot) } + \frac{1}{12} \text{ (figure 8 with a circle and a dot) } \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{48} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{16} \text{diagram} \\
& + \frac{1}{4} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{4} \text{diagram} \\
& + \frac{1}{2} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{12} \text{diagram} \\
& + \frac{1}{8} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{12} \text{diagram} \\
& + \frac{1}{8} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{4} \text{diagram} \\
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& + \frac{1}{4} \text{diagram} + \frac{1}{8} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{3} \text{diagram} \\
& + \frac{1}{24} \text{diagram} + \frac{1}{4} \text{diagram} + \frac{1}{16} \text{diagram} + \frac{1}{4} \text{diagram}
\end{aligned}$$

3.4 One-vertex decomposability

For some discrepancies, the contribution of a bosonic part of a diagram that consists of two pieces connected by *only one* vertex, is equal to the product of the contribution of those pieces. Such diagrams we call *one-vertex reducible*, and discrepancies with this property we call *one-vertex decomposable*. Examples of such discrepancies are those for which \mathcal{B} is translation invariant, i.e., $\mathcal{B}(x, y) = \mathcal{B}(x + a, y + a) \forall x, y, a \in \mathbf{K}$, such as the Fourier diaphony. Also the Lego discrepancy with equal bins is one-vertex decomposable. In contrast, the L_2^* -discrepancy is not one-vertex decomposable.

As a result of the one-vertex decomposability, many diagrams cancel or give zero. For example, the first and the second diagram in (38) cancel, and the fourth gives zero, so that

$$\frac{1}{N} W_1(z) = \frac{1}{4} \text{diagram} + \frac{1}{12} \text{diagram} . \quad (40)$$

To second order, only the following remains:

$$\begin{aligned}
\frac{1}{N^2} W_2(z) = & \frac{1}{48} \text{diag}_1 + \frac{1}{24} \text{diag}_2 + \frac{1}{8} \text{diag}_3 + \frac{1}{16} \text{diag}_4 \\
& + \frac{1}{8} \text{diag}_5 + \frac{1}{4} \text{diag}_6 + \frac{1}{4} \text{diag}_7 + \frac{1}{2} \text{diag}_8 \\
& + \frac{1}{16} \text{diag}_9 + \frac{1}{8} \text{diag}_{10} + \frac{1}{4} \text{diag}_{11} + \frac{1}{12} \text{diag}_{12} + \frac{1}{3} \text{diag}_{13} . \quad (41)
\end{aligned}$$

We now derive a general rule of diagram cancellation. First, we extend the notion of one-vertex reducibility to complete diagrams, including the fermionic part, with the rule that the two pieces both must contain a bosonic part. Consider the following diagram

$$\text{diag}_{42} \quad . \quad (42)$$

The only restriction we put on the “leave” A is that it must be one-vertex irreducible with respect to the vertex that connects it to the fermion loop. For the rest, it may be anything. We define the contribution of the leave by the contribution of the whole diagram divided by $-N$, and denote it with $C(A)$. This contribution includes internal symmetry factors. Now consider a diagram consisting of a fermion loop as in diagram (42) with attached to the one vertex n_1 leaves of type A_1 , n_2 leaves of type A_2 , and so on, up to n_p leaves of type A_p . The extra symmetry factor of such a diagram is $(n_1!n_2!\cdots n_p!)^{-1}$, and, for one-vertex decomposable discrepancies, the contribution is equal to the product of the contributions of the leaves, so that the total contribution is given by

$$-N \prod_{q=1}^p \frac{C(A_q)^{n_q}}{n_q!} . \quad (43)$$

Now we sum the contribution of all possible diagrams of this kind that can be made with the p leaves, and denote the result by

$$\text{diag}_{44} = -N \sum_{n_1, n_2, \dots \geq 1} \prod_{q=1}^p \frac{C(A_q)^{n_q}}{n_q!} = -N \left(e^{\sum_{q=1}^p C(A_q)} - 1 \right) . \quad (44)$$

Because the black square in l.h.s. of Eq. (44) represents all possible ways to put the leaves together onto one vertex, the sum of all possible ways to put the leaves onto one fermion loop is given by

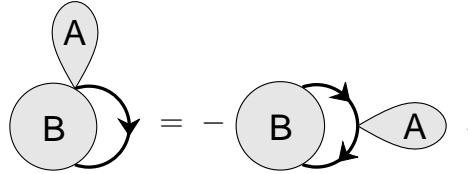
$$\text{diag}_{45} + \text{diag}_{46} + \text{diag}_{47} + \dots = -N \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(e^{\sum_{q=1}^p C(A_q)} - 1 \right)^n . \quad (45)$$

The $(-1)^{n-1}$ in the sum comes from the vertices and $1/n$ is the extra symmetry factor of such diagram with n vertices. The sum can be evaluated further and is equal to

$$-N \log e^{\sum_{q=1}^p C(A_q)} = -N \sum_{q=1}^p C(A_q) , \quad (46)$$

i.e., the sum of all possible ways to put p different leaves onto one fermion loop is equal to the sum of all leaves, each of them put onto its own fermion loop. This means that diagrams, consisting of two or more leaves put onto one fermion loop, cancel.

Now consider the following equation, which holds for every one-vertex decomposable discrepancy:



$$\text{Diagram 1} = - \text{Diagram 2} , \quad (47)$$

where we only assume that B is not of the type on the l.h.s. of Eq. (45). The minus sign comes from the fact that the first diagram has one vertex less. Because the number of fermion lines, a fermion loop consists of is equal to the number of vertices it contains, we can always pair the diagrams into one diagram of the l.h.s. type and one of the r.h.s. type so that they cancel. We can summarize the result with the rule that *for one-vertex decomposable discrepancies, only the one-vertex irreducible diagrams contribute*.

4 Applications

We apply the general formulae given above to the Lego discrepancy, the L_2^* -discrepancy in one dimension and the Fourier diaphony in one dimension.

4.1 The Lego discrepancy

For the Lego problem class, for example defined in [4], the basis consists of a set of characteristic functions ϑ_n , $n = 1, 2, \dots, M$ of M disjunct subspaces of \mathbf{K} . We will denote the measure $\int_{\mathbf{K}} \vartheta_n(x) dx$ of subspace n by w_n and we have $\sum_{n=1}^M w_n = 1$. The strengths σ_n are equal to $1/\sqrt{w_n}$, and the propagator is given by

$$\mathcal{B}(x, y) = \sum_{n=1}^M \frac{\vartheta_n(x) \vartheta_n(y)}{w_n} - 1 . \quad (48)$$

With this choice of σ_n , the discrepancy is just the χ^2 statistic that determines how well the points are distributed over the bins. It is easy to see that $\mathcal{B}_p(x, y) = \mathcal{B}(x, y)$, $p = 2, 3, \dots$, so that the dressed propagator is given by

$$\mathcal{G}_z(x, y) = \frac{1}{1 - 2z} \mathcal{B}(x, y) . \quad (49)$$

In [4], the propagator is given as an $M \times M$ -matrix with matrix elements

$$\mathcal{G}_{n,m}^{(z)} = \frac{1}{1-2z} \left[\frac{\delta_{n,m}}{w_n} - 1 \right] \quad (50)$$

and acting in the M -dimensional function space, rather than as a two-point function. This follows naturally from the path integral, which is an M -dimensional integral. The obvious and correct relation between the two is that

$$\mathcal{G}_z(x, y) = \sum_{n,m=1}^M \vartheta_n(x) \mathcal{G}_{n,m}^{(z)} \vartheta_m(y) . \quad (51)$$

The zeroth order term can be found with the relation of Eq. (37), which results in the following expression

$$W_0(z) = -\frac{1}{2} \log(1-2z) \int_{\mathbf{K}} \mathcal{B}(x, x) dx = -\frac{M-1}{2} \log(1-2z) , \quad (52)$$

in agreement with Eq. (44) in [4]. To write down the first order term, we introduce

$$M_2 = \sum_{n=1}^M \frac{1}{w_n} , \quad \text{and} \quad \eta(z) = \frac{2z}{1-2z} , \quad (53)$$

so that

$$W_1(z) = \frac{1}{8} (M_2 - M^2 - 2M + 2) \eta(z)^2 + \frac{1}{24} (5M_2 - 3M^2 - 6M + 4) \eta(z)^3 . \quad (54)$$

If the bins are equal, so that $w_n = 1/M$ $n = 1, 2, \dots, M$, then only the contribution of the diagrams of Eq. (42) remains, and the result is

$$W_1(z) = -\frac{1}{4} E \eta(z)^2 + \frac{1}{12} (E^2 - E) \eta(z)^3 , \quad (55)$$

where we denote

$$E = M - 1 . \quad (56)$$

To second order in $1/N$, the contribution comes from the diagrams in Eq. (41), and is given by

$$\begin{aligned} W_2(z) = & (5E^3 - 12E^2 + 7E) \frac{\eta(z)^6}{48} + (E^3 - 6E^2 + 5E) \frac{\eta(z)^5}{8} \\ & + (E^3 - 28E^2 + 43) \frac{\eta(z)^4}{48} + (-E^2 - 5E) \frac{\eta(z)^3}{12} . \end{aligned} \quad (57)$$

In Appendix A, we present the expansion of $G(z)$ in the case of equal bins, up to and including the $1/N^4$ term. It is calculated using the path integral expression (3) of $G(z)$ and computer algebra. The reader may check that this expression for $G(z)$ and the above terms of $W(z)$ satisfy $G(z) = e^{W(z)}$ up to the order of $1/N^2$.

4.2 The L_2^* -discrepancy

For the L_2^* -discrepancy in one dimension, for example defined in [4], the gauge freedom is a freedom in the boundary conditions which the members of the problem class have to satisfy. Λ acts on the members as

$$(\Lambda\phi)(x) = -\frac{d^2\phi}{dx^2}(x) , \quad (58)$$

and in the Landau gauge, the boundary conditions are given by

$$\int_{\mathbf{K}} \phi(x) dx = 0 , \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(1) = 0 . \quad (59)$$

The basis in the Landau gauge is given by the set of eigenfunctions of Λ with the boundary conditions above, which is $\{\sqrt{2}\cos(n\pi x), n = 1, 2, \dots\}$, so that the propagator is given by

$$\mathcal{B}(x, y) = \sum_{n=1}^{\infty} \frac{2\cos(n\pi x)\cos(n\pi y)}{n^2\pi^2} = \min(x, y) - x + \frac{1}{2}x^2 - y + \frac{1}{2}y^2 + \frac{1}{3} . \quad (60)$$

The dressed propagator is given by

$$\mathcal{G}_z(x, y) = \sum_{n=1}^{\infty} \frac{2\cos(n\pi x)\cos(n\pi y)}{n^2\pi^2 - 2z} \quad (61)$$

$$= \frac{1}{u^2} - \frac{1}{2u\sin u} \{\cos[u(1 - |x + y|)] + \cos[u(1 - |x - y|)]\} , \quad (62)$$

with

$$u = \sqrt{2z} . \quad (63)$$

The zeroth order term can be obtained using Eq. (37):

$$W_0(z) = -\frac{1}{2} \log \left(\frac{\sin u}{u} \right) , \quad (64)$$

which is the well-known result. After some algebra, also the first order term follows:

$$W_1(z) = \frac{1}{288} \left(24 - 8\frac{u}{\sin u} - 7\frac{u^2}{\sin^2 u} - 7\frac{u}{\tan u} - 2\frac{u^2}{\tan^2 u} \right) . \quad (65)$$

4.3 The Fourier diaphony

Usually, the Fourier diaphony is defined in terms of a basis that is in the Landau gauge already. It is, for example, given in [8], and in one dimension, the basis is given by the functions

$$u_{2n-1}(x) = \sqrt{2}\sin(2\pi nx) , \quad u_{2n}(x) = \sqrt{2}\cos(2\pi nx) , \quad n = 1, 2, \dots , \quad (66)$$

and for the strengths we take

$$\sigma_{2n-1} = \sigma_{2n} = \frac{1}{n} \ , \quad n = 1, 2, \dots \ . \quad (67)$$

Notice that the basis functions satisfy $-\frac{d^2}{dx^2}u_n(x) = \frac{4\pi^2}{\sigma_n^2}u_n(x)$, so that, from this point of view, the only relevant difference between the L_2^* -discrepancy and the Fourier diaphony are the boundary conditions on the members of the problem class.

The propagator is given by

$$\mathcal{B}(x, y) = \sum_{n=1}^{\infty} \frac{2 \cos(2n\pi\{x-y\})}{n^2} = \frac{\pi^2}{3} [1 - 6\{x-y\}(1 - \{x-y\})] \ , \quad (68)$$

where we use the notation $\{x\} = x \bmod 1$. The dressed propagator is given by

$$\mathcal{G}_z(x, y) = \sum_{n=1}^{\infty} \frac{2 \cos(2n\pi\{x-y\})}{n^2 - 2z} = \frac{\pi^2}{v^2} \left(1 - \frac{v \cos[v(2\{x-y\} - 1)]}{2 \sin v} \right) \ , \quad (69)$$

where

$$v = \sqrt{2\pi^2 z} \ . \quad (70)$$

This two-point function is, apart of a factor π^2/v^2 , the same as the one in Eq. (26) in [6]. The zeroth order term can easily be obtained from the dressed propagator and is given by

$$W_0(z) = -\log \left(\frac{\sin v}{v} \right) \ , \quad (71)$$

which is in correspondence with Eq. (21) in [6]. Because the propagator is translation invariant, i.e., $\mathcal{B}(x+a, y+a) = \mathcal{B}(x, y) \ \forall x, y, a \in \mathbf{K}$, the contributions of the first two diagrams in Eq. (38) cancel, and the contribution of the fourth diagram is zero. The contribution of the remaining diagrams gives

$$W_1(z) = \frac{1}{36} \left(3 + v^2 - 3 \frac{v^2}{\sin^2 v} \right) \ . \quad (72)$$

5 Conclusions

In addition to the machinery of QFT introduced in [4], we introduced fermions to calculate the moment generating function G of the probability distribution under sets of random points of a quadratic discrepancy D_N . They allow for an expansion in the inverse of the number of points N of the logarithm W of G , where the contribution to each term in the expansion can be represented by a finite number of connected Feynman diagrams. We have presented the diagrams up to the order of $1/N^2$ for the general case, and derived a

rule of diagram cancellation in the case of special discrepancies, which we call one-vertex decomposable.

We have applied the formalism to the Lego discrepancy, the L_2^* -discrepancy in one dimension and the Fourier diaphony in one dimension, and calculated the first two terms W_0 and W_1/N in the expansion. For the Lego discrepancy, this resulted in Eq. (52) and Eq. (54), for the L_2^* -discrepancy in Eq. (64) and Eq. (65), and for the Fourier diaphony in Eq. (71) and Eq. (72). The Fourier diaphony and the Lego discrepancy with equal binning are one-vertex decomposable. For the latter, we also calculated the term W_2/N^2 , which is in correspondence with the result of an alternative calculation up to the order of $1/N^4$, given in Appendix A.

Calculations become very cumbersome for high orders because of the number of diagrams involved. A situation in which the formalism can still be powerful is when another parameter in the definition of the discrepancy, such as the dimension of the integration region or the number of bins in case of the Lego discrepancy, becomes large. This parameter can then serve as an extra order parameter in the determination of the importance of the contribution of the diagrams, and can lead to a substantial reduction in the number of relevant diagrams. In [9], we will present our results with respect to this for the Lego discrepancy.

Appendix A

If we define, for the Lego-discrepancy with equal bins, $E = M - 1$, $\eta(z) = 2z/(1 - 2z)$ and

$$(1 - 2z)^{E/2} G(z) = \sum_{n,p \geq 0} \frac{\eta(z)^p}{N^n} C_n^{(p)}(E) \quad , \quad (73)$$

then the only non-zero $C_n^{(p)}(E)$ up to $n = 4$ are given by

$$\begin{aligned} C_1^{(2)}(E) &= -\frac{1}{4}E \\ C_1^{(3)}(E) &= E \left(\frac{1}{12}E - \frac{1}{12} \right) \\ C_2^{(3)}(E) &= E \left(-\frac{1}{12}E + \frac{5}{12} \right) \\ C_2^{(4)}(E) &= E \left(\frac{1}{48}E^2 - \frac{53}{96}E + \frac{43}{48} \right) \\ C_2^{(5)}(E) &= E \left(\frac{5}{48}E^2 - \frac{35}{48}E + \frac{5}{8} \right) \\ C_2^{(6)}(E) &= E \left(\frac{1}{288}E^3 + \frac{7}{72}E^2 - \frac{71}{288}E + \frac{7}{48} \right) \\ C_3^{(4)}(E) &= E \left(-\frac{1}{48}E^2 + \frac{7}{12}E - \frac{61}{48} \right) \end{aligned}$$

$$\begin{aligned}
C_3^{(5)}(E) &= E \left(\frac{1}{240}E^3 - \frac{17}{30}E^2 + \frac{583}{120}E - \frac{1451}{240} \right) \\
C_3^{(6)}(E) &= E \left(\frac{53}{576}E^3 - \frac{1153}{384}E^2 + \frac{7423}{576}E - \frac{527}{48} \right) \\
C_3^{(7)}(E) &= E \left(\frac{1}{576}E^4 + \frac{461}{1152}E^3 - \frac{6581}{1152}E^2 + \frac{8663}{576}E - \frac{467}{48} \right) \\
C_3^{(8)}(E) &= E \left(\frac{11}{1152}E^4 + \frac{85}{144}E^3 - \frac{5125}{1152}E^2 + \frac{1555}{192}E - \frac{17}{4} \right) \\
C_3^{(9)}(E) &= E \left(\frac{1}{10368}E^5 + \frac{29}{3456}E^4 + \frac{955}{3456}E^3 - \frac{12475}{10368}E^2 + \frac{953}{576}E - \frac{53}{72} \right) \\
C_4^{(5)}(E) &= E \left(-\frac{1}{240}E^3 + \frac{37}{80}E^2 - \frac{337}{80}E + \frac{1397}{240} \right) \\
C_4^{(6)}(E) &= E \left(\frac{1}{1440}E^4 - \frac{349}{960}E^3 + \frac{7193}{720}E^2 - \frac{15283}{320}E + \frac{67021}{1440} \right) \\
C_4^{(7)}(E) &= E \left(\frac{49}{960}E^4 - \frac{29069}{5760}E^3 + \frac{372169}{5760}E^2 - \frac{571727}{2880}E + \frac{21503}{144} \right) \\
C_4^{(8)}(E) &= E \left(\frac{13}{23040}E^5 + \frac{13979}{23040}E^4 - \frac{2290601}{92160}E^3 + \frac{1446743}{7680}E^2 \right. \\
&\quad \left. - \frac{9583187}{23040}E + \frac{294773}{1152} \right) \\
C_4^{(9)}(E) &= E \left(\frac{73}{6912}E^5 + \frac{35077}{13824}E^4 - \frac{781079}{13824}E^3 + \frac{993515}{3456}E^2 - \frac{564301}{1152}E + \frac{24607}{96} \right) \\
C_4^{(10)}(E) &= E \left(\frac{1}{13824}E^6 + \frac{139}{3072}E^5 + \frac{162721}{34560}E^4 - \frac{596467}{9216}E^3 \right. \\
&\quad \left. + \frac{1653251}{6912}E^2 - \frac{253799}{768}E + \frac{145199}{960} \right) \\
C_4^{(11)}(E) &= E \left(\frac{17}{41472}E^6 + \frac{895}{13824}E^5 + \frac{55025}{13824}E^4 - \frac{1505645}{41472}E^3 \right. \\
&\quad \left. + \frac{19783}{192}E^2 - \frac{137875}{1152}E + \frac{1565}{32} \right) \\
C_4^{(12)}(E) &= E \left(\frac{1}{497664}E^7 + \frac{11}{31104}E^6 + \frac{2431}{82944}E^5 + \frac{155735}{124416}E^4 \right. \\
&\quad \left. - \frac{3942431}{497664}E^3 + \frac{249239}{13824}E^2 - \frac{250141}{13824}E + \frac{2575}{384} \right)
\end{aligned}$$

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